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Sets non-thin at ∞ in $\mathbb{C}^m$

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ABSTRACT

In this paper we define the notion of non-thin at ∞ as follows: Let E be a subset of $\mathbb{C}^m$. For any $R>0$ define $E_R = E \cap \{z \in \mathbb{C}^m\colon\ |z| \leqslant R\}$. We say that E is non-thin at ∞ if

$$\lim_{R\to\infty} V_{E_R}(z) = 0$$

for all $z \in \mathbb{C}^m$, where V_E is the pluricomplex Green function of E. This definition of non-thinness at ∞ has good properties: If $E \subset \mathbb{C}^m$ is non-thin at ∞ and A is pluripolar then $E \setminus A$ is non-thin at ∞; if $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$ are arbitrary sets, then E and F are non-thin at ∞ iff $E \times F \subset \mathbb{C}^m \times \mathbb{C}^n$ is non-thin at ∞ (see Lemma 2). The results of this paper extend some results in [J. Muller, A. Yavrian, On polynomials sequences with restricted growth near infinity, Bull. London Math. Soc. 34 (2002) 189–199] and [Dang Duc Trong, Tuyen Trung Truong, The growth at infinity of a sequence of entire functions of bounded orders, Complex Var. Elliptic Equ. 53 (8) (2008) 717–743].

1. Introduction

Fix $m \in \mathbb{N}$, let $\mathbb{C}^m$ be the usual m-dimensional complex Euclidean space. Before going into the main points, we recall some facts about the potential theory in $\mathbb{C}^m$. Let U be an open subset of $\mathbb{C}^m$. A function $u : U \to [-\infty, \infty)$ is called PSH in U (written $u \in PSH(U)$) if u is upper-semicontinuous and when restricted to any complex line $L \simeq \mathbb{C}$ then u is subharmonic (see [4]).

A $PSH(\mathbb{C}^m)$ function u is said to be of minimal growth if

$$u(z) - \log|z| \leqslant O(1), \quad \text{as } |z| \to \infty,$$

here $|z|$ is the usual Euclidean norm of an element $z \in \mathbb{C}^m$. We denote the set of all such functions by $\mathbb{L}$.

Let E be a subset in $\mathbb{C}^m$. Then the pluricomplex Green function of the set E with pole at infinity (see [4, p. 184]) is

$$V_E(z) = \sup\bigl\{u(z)\colon\ u \in \mathbb{L},\ u|_E \leqslant 0\bigr\} \quad \bigl(z \in \mathbb{C}^m\bigr).$$

V_E is also called the Siciak extremal function of the set E. By definition we see that $V_E \geqslant 0$.

For any function $u : \mathbb{C}^m \to [-\infty, \infty)$ we define its upper-semicontinuous regularization u^* (see [4]) by

$$u^*(z) = \limsup_{\zeta \to z} u(\zeta).$$

Let $V_E^*(z)$ be the regularization of the Green function $V_E(z)$. It is well known (see [4]) that there are only three cases:

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Case 1: $V_E^*(z) \equiv +\infty$. Then E is pluripolar.
Case 2: $V_E^*(z) \in \mathbb{L}$, and $V_E^* \not\equiv 0$.
Case 3: $V_E^*(z) \equiv 0$.

When $m = 1$ then $V_E^*(z) \equiv 0$ iff the set E is non-thin at ∞, or equivalently the set $E^* = \{z\colon\ 1/z \in E\}$ is non-thin at 0 (see [7,2]). Recall that a subset E of $\mathbb{C}^m$ is pluri-thin (or thin for brevity) at a point $a \in \mathbb{C}^m$ if either a is not a limit point of E or there is a neighborhood U of a and a function $u \in PSH(U)$ such that

$$\limsup_{z \to a,\, z \in E \setminus a} u(z) < u(a).$$

If E is not thin then it is called non-thin.

In [7] the authors proved the following result (see also [2, p. 270]).

Proposition 1. *Let E be a closed subset of $\mathbb{C}$. Then the following four statements are equivalent*:

1. *E is non-thin at ∞.*
2. *For any $z \in \mathbb{C}$ we have*

$$\lim_{R \to \infty} V_{E_R}(z) = 0,$$

where we define $E_R = E \cap \{z \in \mathbb{C}\colon\ |z| \leqslant R\}$.

3. *If sequences (P_n) of polynomials and $k_n \geqslant \deg(P_n)$ satisfy*

$$\limsup_{n \to \infty} \left|P_n(z)\right|^{1/k_n} \leqslant 1$$

for all $z \in E$, then

$$\limsup_{n \to \infty} \left|P_n(z)\right|^{1/k_n} \leqslant 1$$

for all $z \in \mathbb{C}$.

4. *$V_E(z) \equiv 0$.*

In $\mathbb{C}^m$ ($m > 1$), we cannot reduce the definition of non-thinness of E at ∞ to the non-thinness at 0 of some other sets E^*, as it was when $m = 1$. However, Proposition 1 suggests a way to define non-thinness at ∞ in higher dimensions:

Definition 1. Let E be a subset of $\mathbb{C}^m$. For any $R > 0$ define $E_R = E \cap \{z \in \mathbb{C}^m\colon\ |z| \leqslant R\}$. We say that E is non-thin at ∞ if

$$\lim_{R \to \infty} V_{E_R}(z) = 0$$

for all $z \in \mathbb{C}^m$.

If E is not non-thin at ∞ then we say E is thin at ∞.

This definition of non-thinness at ∞ has good properties: If $E \subset \mathbb{C}^m$ is non-thin at ∞ and A is pluripolar then $E \setminus A$ is non-thin at ∞; if $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$ are arbitrary sets, then E and F are non-thin at ∞ iff $E \times F \subset \mathbb{C}^m \times \mathbb{C}^n$ is non-thin at ∞ (see Lemma 2).

Our first result in this paper is the following, which is an analog to the case $m = 1$:

Theorem 2. *Let E be a closed subset of $\mathbb{C}^m$. Then the following two statements are equivalent*:

1. *E is non-thin at ∞.*
2. *If sequences (P_n) of polynomials and $k_n \geqslant \deg(P_n)$ satisfy*

$$\limsup_{n \to \infty} \left|P_n(z)\right|^{1/k_n} \leqslant 1$$

for all $z \in E$, then

$$\limsup_{n \to \infty} \left|P_n(z)\right|^{1/k_n} \leqslant 1$$

for all $z \in \mathbb{C}^m$.

If E is non-thin at ∞ then it is easy to see that $V_E \equiv 0$. The converse is not true for $m \geqslant 2$, by the following example (which essentially is the same as Example 1.1 in [2]).

Example 1. Let E be a subset of $\mathbb{C}^2$ defined by

$$E = \{(z_1, z_2) \in \mathbb{C}^2\colon\ |z_2| \leqslant 1\} \cup \{(z_1, z_2) \in \mathbb{C}^2\colon\ z_1 = 0\}.$$

Then $V_E \equiv 0$ but E is thin at ∞.

Proof. Using arguments in Example 1.1 in [2], it is easy to see that $V_E \equiv 0$.

If E was non-thin at ∞ then since $A = \{(z_1, z_2) \in \mathbb{C}^2\colon\ z_1 = 0\}$ is pluripolar, by Lemma 2 we should have $E \setminus A$ is non-thin at ∞. In particular we should have $V_{E\setminus A} \equiv 0$. However as computed in [2] we have $V_{E\setminus A}(z) = \log^+ |z_2|$, which is a contradiction.

Hence E is thin at ∞. □

The following result gives a characterization of sets E with $V_E \equiv 0$, which also shows that the non-thin at ∞ sets are not rare.

Theorem 3. *Let E be a closed subset of $\mathbb{C}^m$. Then $V_E \equiv 0$ iff any open neighborhood of E is non-thin at ∞.*

Example 2. Applying Theorem 3 to Example 1, we see that the set

$$\{(z_1, z_2) \in \mathbb{C}^2\colon\ |z_2| \leqslant 1\} \cup W,$$

is non-thin at ∞ in $\mathbb{C}^2$, where W is an open neighborhood of the set $\{(z_1, z_2) \in \mathbb{C}^2\colon\ z_1 = 0\}$.

Example 3. Let $\triangle$ be a collection of complex lines L in $\mathbb{C}^m$ such that

$$\bigcup_{L\in\triangle} L = \mathbb{C}^m.$$

Let E be a closed subset of $\mathbb{C}^m$ such that for each $L \in \triangle$ the set $E \cap L$ considered as a subset in the one-dimensional complex line L is non-thin at ∞. Then E is non-thin at ∞ as a subset in $\mathbb{C}^m$.

Proof. By Theorem 2 we need only to show that: If (P_n) is a sequence of polynomials, and $k_n \geqslant \deg(P_n)$ such that

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leqslant 1$$

for $z \in E$, then

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leqslant 1$$

for all $z \in \mathbb{C}^m$.

Let $w \in \mathbb{C}$ be a coordinate for L such that the coordinates $z_1, \ldots, z_n$ of $\mathbb{C}^m$ are linear functions of w when restricted on L, and denote by $P_{n,L}(w)$ the restriction of P_n to L. Then $P_{n,L}(w)$ is a polynomial in one complex variable w of degree $\deg(P_{n,L}(w)) \leqslant \deg(P_n) \leqslant k_n$.

Then since

$$\limsup_{n\to\infty} |P_{n,L}(w)|^{1/k_n} \leqslant 1$$

for $w \in E \cap L$, and $E \cap L$ is non-thin at ∞ in the complex line L, hence

$$\limsup_{n\to\infty} |P_n(w)|^{1/k_n} \leqslant 1$$

for all $w \in L$. Since this is true for any complex line $L \in \triangle$, it is also true for their union, which is equal to $\mathbb{C}^m$. □

In Theorem 2 of [3], we prove a Phragmen–Lindelof type inequality for subsets of $\mathbb{C}$ having "big" capacity (by "big" capacity we mean a set E for which $C(E_R)$ grows proportionally to $\log R$). In the sequel, we state and prove a similar result for higher dimensions (see the remarks at the end of this section for more discussion about this).

Let $K \subset \mathbb{C}^m$ be compact. Then the Robin constant of K is defined as (see [6])

$$\gamma(K) = \limsup_{z\to\infty} [V_E^*(z) - \log|z|],$$

and the $\mathbb{C}^m$-capacity of K is

$$C(K) = e^{-\gamma(K)}. \tag{1.1}$$

Let $K \subset \{z \in \mathbb{C}^m\colon |z| \leqslant s\}$ be compact and non-pluripolar, where $s > 0$. Then for any $0 < t < 1$ we have

$$\sup_{|z|\leqslant s} V_K^*(z) \leqslant m\frac{1+t}{(1-t)^{2m-1}}\left(\log s + \log\frac{1}{t} - \log C(K)\right). \tag{1.2}$$

In fact, the proof of theorem in [8, p. 319] gives that

$$\int\limits_{S^{2m-1}} V_K^*(rz)\,d\sigma(z) \leqslant m\bigl(\log r - \log C(K)\bigr),$$

for all $r \geqslant s$, where $d\sigma$ is the normalized area measure on the unit sphere S^{2m-1} in $\mathbb{C}^m$. Since V_K^* is a non-negative PSH function, if in the above inequality we choose $r = s/t$ and apply Harnack inequality (see for example Chapter 3 in [1]), we obtain

$$\sup_{|z|\leqslant s} V_K^*(z) \leqslant \frac{1+t}{(1-t)^{2m-1}}\int\limits_{S^{2m-1}} V_K^*(sz/t)\,d\sigma(z) \leqslant m\frac{1+t}{(1-t)^{2m-1}}\left(\log s + \log\frac{1}{t} - \log C(K)\right).$$

Lemma 1. *Let $E \subset \mathbb{C}^m$ be closed and satisfy the following condition: there exists $0 < \beta \leqslant 1$ such that*

$$\limsup_{R\to\infty}\frac{\log C(E_R)}{\log R} \geqslant \beta > 1 - \frac{1}{m}, \tag{1.3}$$

where $C(E_R)$ is determined from formula (1.1).

Let (P_n) be a sequence of polynomials and $k_n \geqslant \deg(P_n)$. Assume that for all $z \in E$ we have

$$\limsup_{n\to\infty}\bigl|P_n(z)\bigr|^{1/k_n} \leqslant h\bigl(|z|\bigr)$$

where h is a positive real function such that

$$\limsup_{R\to\infty}\frac{\log h(R)}{\log R} \leqslant \tau < \infty. \tag{1.4}$$

For all $\lambda \in S^{2m-1}$, we define

$$\exp\left\{\limsup_{n\to\infty}\frac{1}{2\pi k_n}\int\limits_0^{2\pi}\log\bigl|P_n\bigl(e^{i\theta}\lambda\bigr)\bigr|\,d\theta\right\} =: C_\lambda.$$

Then for any $w \in \mathbb{C}$ and $\lambda \in S^{2m-1}$ we have

$$\limsup_{n\to\infty}\bigl|P_n(w\lambda)\bigr|^{1/k_n} \leqslant C_\lambda\bigl(1+|w|\bigr)^{\tau/[1-m(1-\beta)]}.$$

Corollary 1. *Let $E \subset \mathbb{C}^m$ be closed. Then E is non-thin at ∞ if it satisfies the following condition: There exists $0 < \beta \leqslant 1$ such that*

$$\limsup_{R\to\infty}\frac{\log C(E_R)}{\log R} \geqslant \beta > 1 - \frac{1}{m},$$

where $C(E_R)$ is determined from formula (1.1).

Corollary 2. *Let Δ be a non-empty closed subset of $\mathbb{P}^{m-1}$ and $F = \bigcup_{\lambda\in\Delta} L_\lambda$ where L_λ is the complex line going through 0 with direction λ. Assume that*

$$\limsup_{R\to\infty}\frac{\log C(F_R)}{\log R} = \gamma > 1 - \frac{1}{m},$$

where as before $F_R = F \cap \{z \in \mathbb{C}^m\colon |z| \leqslant R\}$.

Let $E \subset \mathbb{C}^m$ be closed and satisfy the following condition: there exists a constant $0 < \beta \leqslant 1$ with

$$m(1-\gamma) < \beta$$

such that

$$\liminf_{s\to\infty}\frac{1}{\log s}\frac{1}{2\pi}\int\limits_0^{2\pi} V_{E_s}^*\bigl(se^{it}\lambda\bigr)\,dt \leqslant 1 - \beta \tag{1.5}$$

for all $\lambda \in \Delta$. Then E is non-thin at ∞.

Interesting questions may arise such as whether we have a Wiener criterion for non-thinness at ∞ in $\mathbb{C}^m$, or can we extend the above results to sequences of entire functions of bounded orders as done in [3] for one-dimensional. An example in [5], giving lower bounds for a constant in another result of Taylor in the cited paper [8], suggests that the best lower bound for β in Lemma 1 and Corollary 1 may be $1-1/m$. If the previous claim is true, it means that there are sets in $\mathbb{C}^m$ with "big" capacities yet being thin at ∞ when $m>1$. We hope to return to these issues in a future paper.

The rest of this paper is devoted to proving the results stated above.

2. Proofs of the results

Lemma 2.

(a) *Let E be a subset of $\mathbb{C}^m$. If E is non-thin at ∞ and A is pluripolar then $E\setminus A$ is non-thin at ∞.*
(b) *Let $E\subset\mathbb{C}^m$ and $F\subset\mathbb{C}^n$ be arbitrary sets. Then $E\times F\subset\mathbb{C}^m\times\mathbb{C}^n$ is non-thin at ∞ iff $E\subset\mathbb{C}^m$ and $F\subset\mathbb{C}^n$ are non-thin at ∞.*

Proof. (a) Define $F=E\setminus A$. For any $R>0$ the set $E_R=E\cap\{z\in\mathbb{C}^m\colon |z|\leqslant R\}$ is bounded. Hence by Corollary 5.2.5 in [4] we have

$$V^*_{E_R}=V^*_{F_R}.$$

Fix a sequence $R_n\to\infty$, using the same argument as in the proof of Theorem 2 (see below), there exists a pluripolar set B such that

$$\lim_{n\to\infty}V_{F_{R_n}}(z)=\lim_{n\to\infty}V^*_{F_{R_n}}(z)=\lim_{n\to\infty}V^*_{E_{R_n}}(z)=\lim_{n\to\infty}V_{E_{R_n}}(z)=0,$$

for all $z\in\mathbb{C}^m\setminus B$. Since $V_{F_{R_n}}(z)$ are PSH, the previous equality is then also true for $z\in B$, thus proves (a).

(b) Let $R>0$. Then one can check easily that

$$\max\bigl\{V_{E_R}(z),V_{F_R}(w)\bigr\}\leqslant V_{E_R\times F_R}(z,w)\leqslant V_{E_R}(z)+V_{F_R}(w),$$

where $z\in\mathbb{C}^m,\ w\in\mathbb{C}^n$. This completes the proof of case (b). $\square$

Now we proceed to proving Theorem 2.

Proof of Theorem 2. In this proof fix a sequence $R_n\nearrow\infty$.

($2\Rightarrow 1$) Since E is closed, for any $R>0$ we have E_R is compact.

Fix $z_0\in\mathbb{C}^m$. By Siciak's theorem (see Theorem 5.1.7 in [4]) there exists a sequence of polynomials (P_n) of degree (k_n) such that

$$\|P_n\|^{1/k_n}_{E_{R_n}}\leqslant 1,$$

for all $n=1,2,\dots$, and

$$\lim_{n\to\infty}\log\bigl|P_n(z_0)\bigr|^{1/k_n}=\lim_{n\to\infty}V_{E_{R_n}}(z_0). \tag{2.1}$$

Then for any $z\in E$ we have

$$\limsup_{n\to\infty}\bigl|P_n(z)\bigr|^{1/k_n}\leqslant 1.$$

Hence by assumption that Statement 2 is true, we have

$$\limsup_{n\to\infty}\bigl|P_n(z)\bigr|^{1/k_n}\leqslant 1$$

for all $z\in\mathbb{C}^m$. In particular, with $z=z_0$ we get from (2.1) that

$$\lim_{n\to\infty}V_{E_{R_n}}(z_0)\leqslant 0.$$

Since $z_0\in\mathbb{C}^m$ is arbitrary and obviously $V_E(z)\geqslant 0$ for all z, we get Statement 1.

($1\Rightarrow 2$) We use the ideas in [7]. Assume that Statement 1 is true. Consider any sequence (P_n) of polynomials and $k_n\geqslant\deg(P_n)$ such that

$$\limsup_{n\to\infty}\bigl|P_n(z)\bigr|^{1/k_n}\leqslant 1 \tag{2.2}$$

for all $z\in E$. For each n define

$$v_n(z)=\log\bigl|P_n(z)\bigr|^{1/k_n}.$$

Note that Statement 1 implies that for R large enough then E_R is non-pluripolar. Fix $R > 0$ such that E_R is non-pluripolar. For any $h, l \in \mathbb{N}$ define

$$E_R^{h,l} = \bigcap_{n=h}^{\infty} \left\{ z \in E_R: \ v_n(z) < \frac{1}{l} \right\}.$$

Then $E_R^{h,l} \subset E_R^{h+1,l}$ and from (2.2)

$$\bigcup_{h=1}^{\infty} E_R^{h,l} = E_R \tag{2.3}$$

for any $l \in \mathbb{N}$.

By definition of the pluricomplex Green function, for any $h, l \in \mathbb{N}$, $n \geqslant h$ and $z \in \mathbb{C}^m$

$$v_n(z) \leqslant V_{E_R^{h,l}}(z) + \frac{1}{l} \leqslant V^*_{E_R^{h,l}}(z) + \frac{1}{l}.$$

Hence taking the limsup of the above inequality as $n \to \infty$ we get

$$v(z) = \limsup_{n\to\infty} v_n(z) \leqslant V_{E_R^{h,l}}(z) + \frac{1}{l} \leqslant V^*_{E_R^{h,l}}(z) + \frac{1}{l},$$

for all $h, l \in \mathbb{N}$, $z \in \mathbb{C}^m$. Take the limit of this inequality as $h \to \infty$. Using (2.2), we see from Corollary 5.2.6 in [4] that

$$v(z) \leqslant \lim_{h\to\infty} V^*_{E_R^{h,l}}(z) + \frac{1}{l} = V^*_{E_R}(z) + \frac{1}{l},$$

for all $l \in \mathbb{N}$, $z \in \mathbb{C}^m$, $R > 0$. Since l is arbitrary, we get

$$v(z) \leqslant V^*_{E_R}(z), \tag{2.4}$$

for all $z \in \mathbb{C}^m$, $R > 0$.

For each $n \in \mathbb{N}$ define

$$A_n = \left\{ z \in \mathbb{C}^m: \ V_{E_{R_n}}(z) < V^*_{E_{R_n}}(z) \right\}$$

then A_n is pluripolar thus has Lebesgue measure zero, and $V_{E_{R_n}}(z) = V^*_{E_{R_n}}(z)$ for $z \in \mathbb{C}^m \setminus A_n$. Hence

$$A = \bigcup_{n=1}^{\infty} A_n$$

is also pluripolar and of Lebesgue measure zero, and we have $V_{E_{R_n}}(z) = V^*_{E_{R_n}}(z)$ for $z \in \mathbb{C}^m \setminus A$ and $n \in \mathbb{N}$. Hence for $z \in \mathbb{C}^m \setminus A$, applying (2.4) we get

$$v(z) \leqslant \lim_{n\to\infty} V^*_{E_{R_n}}(z) = \lim_{n\to\infty} V_{E_{R_n}}(z) = 0$$

for all $z \in \mathbb{C}^m \setminus A$. Since A is of Lebesgue measure zero, by definition of $v(z)$, we see that $v(z) \leqslant 0$ for all $z \in \mathbb{C}^m$. This completes the proof of Theorem 2. □

Now we prove Theorem 3.

Proof of Theorem 3. ($\Rightarrow$) Let E be a closed subset of $\mathbb{C}^m$. Assume that $V_E \equiv 0$. We will show that any open neighborhood of E is non-thin at ∞. Let F be any open neighborhood of E. Then for any $R > 0$ since E_R is compact, F is open and contains E_R, we can find $1 > \epsilon = \epsilon_R > 0$ such that

$$E_{R,\epsilon} = \left\{ z \in \mathbb{C}^m: \ \mathrm{dist}(z, E_R) \leqslant \epsilon \right\} \subset F.$$

By Corollary 5.1.5 in [4] we have $V^*_{E_{R,\epsilon}}(z) = 0$ for $z \in E_{R,\epsilon} \supset E_R$. Now it is obvious that $E_{R,\epsilon} \subset F_{R+1}$ hence for $z \in E_R$

$$V^*_{F_{R+1}}(z) = 0. \tag{2.5}$$

Define

$$v(z) = \lim_{R\to\infty} V^*_{F_R}(z),$$

then v is the limit of a decreasing sequence of PSH functions hence v is itself PSH. By (2.5) for $z \in E$ we have $v(z) \equiv 0$. Thus by definition of the pluricomplex Green function

$$v(z) \leqslant V_E(z) = 0$$

for all $z \in \mathbb{C}^m$. This shows that F is non-thin at ∞.

($\Leftarrow$) Assume that E is an arbitrary subset of $\mathbb{C}^m$ with $V_E \not\equiv 0$. Then $V_E(z_0) > 0$ for some $z_0 \in \mathbb{C}^m$, hence by definition of the pluricomplex Green function, there exists a function $u \in \mathbb{L}$ such that $u(z) \leqslant 0$ for $z \in E$, and $u(z_0) > 0$. Define

$$F = \{z \in \mathbb{C}^m:\ u(z) < u(z_0)/2\}.$$

F is open because u is upper-semicontinuous, and $E \subset F$ because $u|_E \leqslant 0$ and $u(z_0) > 0$. Now $u(z) < u(z_0)/2$ for $z \in F$, hence we have

$$u(z) \leqslant V_F(z) + u(z_0)/2$$

for all $z \in \mathbb{C}^m$. In particular choose $z = z_0$; we have $V_F(z_0) \geqslant u(z_0)/2 > 0$, hence F is thin at ∞. □

We conclude this section presenting the proofs of Lemma 1 and Corollaries 1 and 2.

Proof of Lemma 1. Define

$$u(z) = \limsup_{n\to\infty} \log\left|P_n(z)\right|^{1/k_n},$$

and let $u^*(z)$ be its upper-semicontinuous regularization. Then $u^*(z) \in \mathbb{L}$.

Now we define

$$\kappa_0 = \limsup_{z\in\mathbb{C}^m,\, z\to\infty} \frac{u^*(z)}{\log|z|}. \tag{2.6}$$

For any $R > 0$, by the definition of the pluricomplex Green function

$$u^*(z) \leqslant \kappa_0 V_{E_R}(z) + \log h(R). \tag{2.7}$$

For any $\lambda \in S^{2m-1}$ define

$$\kappa(\lambda) = \limsup_{w\in\mathbb{C},\, w\to\infty} \frac{u(w\lambda)}{\log|w|} \leqslant \kappa_0.$$

Fix $\lambda \in S^{m-1}$. By definition

$$u(w\lambda) = \limsup_{n\to\infty} \frac{1}{k_n} \log\left|P_n(w\lambda)\right|$$

for all $w \in \mathbb{C}$. Hence using (2.7) and (1.2), for any $s > 0$, $0 < t < 1$ we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{k_n}\log\left|P_n\left(se^{i\theta}\lambda\right)\right| d\theta &\leqslant \frac{1}{2\pi}\int_0^{2\pi} u\left(se^{i\theta}\lambda\right) d\theta \\
&\leqslant \kappa_0 \frac{1}{2\pi}\int_0^{2\pi} V_{E_s}\left(se^{i\theta}\lambda\right) d\theta + \log h(s) \\
&\leqslant \kappa_0 m \frac{1+t}{(1-t)^{2m-1}}\left(\log s + \log\frac{1}{t} - \log C(E_s)\right) + \log h(s).
\end{aligned}$$

By the previous inequality and (1.4), we get for any $0 < t < 1$:

$$\liminf_{s\to\infty} \limsup_{n\to\infty} \frac{1}{\log s}\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{k_n}\log\left|P_n\left(se^{i\theta}\lambda\right)\right| d\theta \leqslant \kappa_0 m \frac{1+t}{(1-t)^{2m-1}}(1-\beta) + \tau. \tag{2.8}$$

Taking the limit $t \to 0^+$ in the previous inequality we obtain

$$\liminf_{s\to\infty} \limsup_{n\to\infty} \frac{1}{\log s}\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{k_n}\log\left|P_n\left(se^{i\theta}\lambda\right)\right| d\theta \leqslant \kappa_0 m(1-\beta) + \tau. \tag{2.9}$$

Then by Lemma 2 in [3] we have

$$\kappa(\lambda) \leqslant \kappa_0 m(1-\beta) + \tau. \tag{2.10}$$

Since (2.10) is true for any $\lambda \in S^{2m-1}$, by definition (2.6) and Lemma 2 in [3] we have

$$\kappa_0 \leqslant \kappa_0 m(1-\beta) + \tau,$$

or equivalently

$$\kappa_0 \leqslant \frac{\tau}{1-m(1-\beta)}.$$

From the above inequality, using Lemma 2 in [3] we get the conclusion of Lemma 1. □

Proof of Corollary 1. Let (P_n) be a sequence of polynomials and let $k_n \geqslant \deg(P_n)$ satisfying

$$u(z) = \limsup_{n\to\infty} \left|P_n(z)\right|^{1/k_n} \leqslant 1$$

for all $z \in E$. Applying Lemma 1 for the set E, the sequences (P_n) and (k_n), and $\tau = 0$, we get

$$\limsup_{n\to\infty} \left|P_n(z)\right|^{1/k_n} \leqslant 1,$$

for all $z \in \mathbb{C}^m$. Then Theorem 2 implies that E is non-thin at ∞. □

Proof of Corollary 2. From the assumptions, we have that E_R is non-pluripolar for $R > 0$ large enough.

Let (P_n) be a sequence of polynomials and let $k_n \geqslant \deg(P_n)$ satisfying

$$u(z) = \limsup_{n\to\infty} \left|P_n(z)\right|^{1/k_n} \leqslant 1$$

for all $z \in E$. Define

$$\kappa_0 = \limsup_{z\in\mathbb{C}^m,\, |z|\to\infty} \frac{u^*(z)}{\log|z|} \leqslant 1,$$

and

$$\kappa_1 = \sup_{\lambda\in\Delta} \limsup_{w\in\mathbb{C},\, |w|\to\infty} \frac{u^*(w\lambda)}{\log|w|} \leqslant \kappa_0.$$

Applying Lemma 1 for the set F, the sequences (P_n) and (k_n), and $\tau = \kappa_1$ we obtain

$$\kappa_0 \leqslant \frac{\kappa_1}{1-m(1-\gamma)}.$$

Using assumption (1.5), apply formula (2.10) for all $\lambda \in \Delta$, for the set E, the sequences (P_n) and (k_n), and $\tau = 0$; we obtain

$$\kappa_1 \leqslant \kappa_0(1-\beta).$$

Hence we have

$$\frac{\kappa_1}{1-\beta} \leqslant \kappa_0 \leqslant \frac{\kappa_1}{1-m(1-\gamma)}.$$

From our assumption that $\beta > m(1-\gamma)$, the above inequality implies that $\kappa_0 = \kappa_1 = 0$. Hence $u^*(z)$ is a constant. Moreover, $u^*(z) \leqslant 0$ on the set E which is non-pluripolar, hence $u^*(z) \leqslant 0$ for all $z \in \mathbb{C}^m$. By Theorem 2, E is non-thin at ∞. □

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